Rarita-Schwinger field in non-abelian Klein-Kaluza theories

1982 J. Phys. A: Math. Gen. 15 2441

(http://iopscience.iop.org/0305-4470/15/8/024)

J. Phys. A: Math. Gen. **15** (1982) 2441–2454. Printed in Great Britain

Rarita–Schwinger field in non-abelian Klein–Kaluza theories

M W Kalinowski

Institute of Philosophy and Sociology, Polish Academy of Sciences, Nowy Swiat 72, 00-330 Warsaw, Poland and
Department of Physics, University of Toronto, Toronto, Ontario, Canada, M5S 1A7

Received 14 December 1981, in final form 9 March 1982

Abstract. In this paper we deal with $\frac{3}{2}$-spinor fields in the framework of the non-abelian Klein–Kaluza theory. We introduce a dimensional reduction procedure for a $\frac{3}{2}$-spinor field and a generalisation of the minimal coupling scheme. We get dipole electric moments for a $\frac{3}{2}$-spin particle of value 10^{-31} cm and *PC* breaking for a gauge group G with odd parameters. Reflections in higher (additional) dimensions were proposed as a conjugation of 'colour' charges connected with Yang–Mills fields. Our approach avoids the Velo–Zwanziger paradox (acausal propagation in an external gauge field).

1. Introduction

In this paper we deal with $\frac{3}{2}$-spinor fields in the framework of non-abelian Klein–Kaluza theories. On an $(n+4)$-dimensional manifold P (metrised fibre bundle) we have introduced one-form spinor fields. These forms are horizontal (in the sense of a connection on the bundle P) and take values from the fundamental representation of the group $SO(1, n+3)$ (Spin $(1, n+3)$).

We assume that this one-form spinor field depends on the group coordinates in a trivial way, i.e. by the action of the group G (G is a gauge group of the Yang–Mills field which we combine with gravity in the Klein–Kaluza framework). We introduce for this one-form spinor field a new kind of gauge derivative. These gauge derivatives were defined in Kalinowski (1981a, b, e) in the five-dimensional (electromagnetic) case and in Kalinowski (1981c) for the $\frac{1}{2}$-spinor field in the non-abelian case. Here we generalise this approach.

Simultaneously we define a dimensional reduction procedure for the one-form spinor field. It contains three steps:

(1) we take a section of the bundle P and apply it for a one-form spinor field Ψ;

(2) we restrict the group $SO(1, n+3)$ to $SO(1, 3)$ for Ψ;

(3) we decompose Ψ to one-form spinor fields with values from the Dirac representation of $SL(2, C)$.

After that we get a tower of $2^{[n/2]}$ one-form spinor fields on a space–time E.

We have treated here the $\frac{3}{2}$-spinor field as a one-form with values in the Dirac representation of $SL(2, C)$ similarly as in Kalinowski (1981b), Isenburg *et al* (1977). In Kalinowski (1981a, b, e) a similar construction for the one-dimensional (electromagnetic) case was introduced. Here we clarify this construction as a kind of dimensional reduction. In Kalinowski (1981b) we presented a similar construction for the $\frac{1}{2}$-spinor field. Next we generalise a minimal coupling scheme for a one-form spinor field Ψ. We define on the $(n+4)$-dimensional manifold a Lagrangian form as in Kalinowski (1981b), Isenburg *et al* (1977). In this Lagrangian we construct a new gauge derivative for the one-form spinor field Ψ. This procedure is a simple generalisation of that of Kalinowski (1981b, e), Isenburg *et al* (1977). In the Lagrangian new terms appear similar to that of Kalinowski (1981b, e), Isenburg (1977). In Kalinowski (1981b, e) such a term was interpreted as the interaction of the dipole electric moment of a fermion with the electromagnetic field. Here the interpretation is more complex. If we perform the dimensional reduction procedure, we get on E (space–time) a sum of Lagrangians for all $\frac{3}{2}$ fermions from a tower describing the interaction of these fermions with gravity and Yang–Mills fields in a classical, already known way, plus new terms. These new terms describe interactions of the Yang–Mills fields with $\frac{3}{2}$-spin fermions from a tower. If the number of group parameters is odd ($\dim G = n = 2l+1$) some of these terms may be interpreted as the interaction of dipole electric moments of fermions with the electromagnetic field. For an even number of parameters of the group G ($\dim G = n = 2l$) such terms are absent. Thus a dipole electric moment of a fermion is possible only in the case of an even number of parameters of the group G. But apart from these terms we also have other terms. These terms may be treated as anomalous dipole moments for 'magnetic' parts of the Yang–Mills field. For an even number of group parameters we have a *PC* breaking. This breaking is obviously very small because the value of the dipole electric moments of $\frac{3}{2}$-spin fermions is about 10^{-31} (cm)q. Similarly as in Kalinowski (1981b, c, e) this value depends on fundamental constants only.

We also consider the Velo–Zwanziger (1969) paradox. It is very well known that a minimal coupling scheme is not well defined for the $\frac{3}{2}$-spinor field. The Rarita–Schwinger equation is relativistic covariant, but solutions are acausal.

The last property is related to the fact that differential consequences for the Rarita–Schwinger equation become algebraic constraints (Velo and Zwanziger 1969). These constraints depend on the strength of the electromagnetic (or Yang–Mills) field. In this paper we generalise the minimal coupling scheme and use the differential forms formalism for the Rarita–Schwinger field. Hence we obtain new terms. These terms are some 'interference' effects due to the gravitational and Yang–Mills fields interacting with the $\frac{3}{2}$-spinor field. The existence of these new terms has important consequences. Due to this, the first differential consequences for the Rarita–Schwinger equation are differential equations. We do not get any algebraic constraints depending on the Yang–Mills field. Hence the Velo–Zwanziger paradox is avoided.

In this paper we also define discrete transformations on P and interpret them as operators of parity, time reversal, charge conjugations, *PC* and *PCT*. Charge conjugations are defined as reflections in n additional dimensions (gauge dimensions).

The paper is organised as follows. In § 2 we describe some elements of the non-abelian Klein–Kaluza theory and define geometric quantities which we use throughout the paper. In § 3 we deal with the dimensional reduction procedure for the $\frac{3}{2}$-spinor field. In § 4 we introduce a generalisation of the minimal coupling scheme for the $\frac{3}{2}$-spinor field. We get new terms in the Lagrangian and interpret them. In § 5 we discuss the Velo–Zwanziger paradox and prove that it is absent in our case. In § 6 we define a discrete transformation on the manifold P. The appendix is devoted to elements of the Clifford algebra which we use in the paper.

Finally we would like to give some remarks concerning § 2 in particular, but also important for the whole paper. We employ the concept of a principal fibre bundle

P, as a base for another principal bundle P'. This is a general idea of the Klein–Kaluza theory. We make P a (pseudo)Riemannian space by defining a metric tensor γ on P. This is equivalent to defining a metric connection (choice of horizontal subspace of $T_u(P')$, $\Lambda_u u \in P'$) in the principal bundle P'—the bundle of linear frames over P as a base. The structure group for P is G (the gauge group for the Yang–Mills field), while the structure group for P' is $\mathrm{GL}(n+4, R)$, reducible to $\mathrm{SO}(1, n+3)$ ($n = \dim \mathrm{G}$). In this case we have two operators of horizontality 'hor' and 'hor_1'. The first is referred to P and the second to P'. Thus this might lead to a misunderstanding. For this reason we use throughout the paper only one operator of horizontality, 'hor', with respect to a connection on P (none on P'!). Simultaneously we use a classical linear metric connection on P as a (pseudo)Riemannian manifold. Due to this all considerations are simpler. Nevertheless there also exists in the theory a third principal bundle P'' with a base E (space–time) and the structural group $\mathrm{GL}(4, R)$ reducible to $\mathrm{SO}(1, 3)$ (Lorentz group). This is the bundle of linear frames over E as a base. Thus appears the third operator of horizontality 'hor_2' referred to P''. For simplicity we also use on E a classical linear metric (Riemannian) connection. From the physical point of view, the author believes that the most important structure here is a gauge structure connecting to the Yang–Mills field, i.e. the principal fibre bundle P. The remaining structures, the metric tensor and linear connections, are some derivations from this fundamental residual structure. In this way the first operator of horizontality 'hor' plays a fundamental role. This concept is developed in the paper.

2. The Klein–Kaluza theory

Let us introduce the principal fibre bundle P over the space–time E with the structural group G and the projection π, and let ω be a connection form on P. Let us suppose that (E, g) is a manifold with a metric tensor g and Riemann connection $\bar{\omega}_{\alpha\beta}$, where $g = g_{\alpha\beta}\bar{\theta}^{\alpha} \otimes \bar{\theta}^{\beta}$. The signature of g is $(---+)$ and $\bar{\theta}^{\alpha}$ is a frame on E. Let us introduce the natural frame on P:

$$\theta^A = (\pi^*(\bar{\theta}^{\alpha}), \theta^a = \lambda\omega^a), \qquad \lambda = \text{constant}. \tag{2.1}$$

$\omega = \omega^a X_a$ is a connection on P. (ω^a are dual to fundamental fields on P.) The two-form of curvature of connection ω is

$$\Omega = \text{hor}\, \mathrm{d}\omega = \tfrac{1}{2} H^a_{\mu\nu}\theta^{\mu} \wedge \theta^{\nu} X_a, \qquad X_a \in \mathfrak{G} \text{ the Lie algebra of G.} \tag{2.2}$$

Ω obeys the structural Cartan equation:

$$\Omega = \mathrm{d}\omega + \tfrac{1}{2}[\omega, \omega]. \tag{2.3}$$

Bianchi's identity for ω is

$$\text{hor}\, \mathrm{d}\Omega = 0. \tag{2.4}$$

The map $e: E \supset U \to P$, so that $e \circ \pi = \text{id}$ is called a cross section. From the physical point of view it means choosing the gauge. Thus

$$\begin{aligned} e^*\omega &= e^*(\omega^a X_a) = A^a_{\mu}\bar{\theta}^{\mu} X_a \\ e^*\Omega &= e^*(\Omega^a X_a) = \tfrac{1}{2} F^a_{\mu\nu}\bar{\theta}^{\mu} \wedge \bar{\theta}^{\nu} X_a \end{aligned} \tag{2.5}$$

where

$$F^a_{\mu\nu} = \partial_{\mu} A^a_{\nu} - \partial_{\nu} A^a_{\mu} + C^a_{bc} A^b_{\mu} A^c_{\nu}. \tag{2.6}$$

X_a, $a=1,2,\ldots,\dim G=n$ are generators of the Lie algebra of the group G, $\mathfrak{G}$, and $[X_a, X_b]=C^c_{ab}X_c$.

A covariant derivation on P with respect to ω, d_1 is defined as follows

$$d_1\Psi = \text{hor}\, d\Psi \qquad \text{(hor is understood in the sense of } \omega\text{)}. \tag{2.7}$$

This derivation is called 'gauge' derivation, where Ψ is for example a one-form field (with values in spinor space) on P. It is convenient to introduce the following notations. Capital Latin indices A, B, C run $1,2,3,4,\ldots,n+4$, $\dim G=n$. Lower case Greek indices α, β, γ, $\delta=1,2,3,4$ and lower case Latin $a,b,c,d=5,6,\ldots,n+4$. The bar over θ^α and $\omega_{\alpha\beta}$ (i.e. $\bar\theta^\alpha$, $\bar\omega_{\alpha\beta}$) indicates that both quantities are defined on E.

Let us now introduce a tensor $\gamma=\gamma_{AB}\theta^A\otimes\theta^B$ on the manifold P in the natural way (Trautman 1970, 1971, 1981). Let $X, Y\in T_{\tan}(P)$.

$$\gamma(X,Y)=g(\pi', X, \pi'Y)+h_{ab}\theta^a(X)\theta^b(Y)$$

or (2.8)

$$\gamma=\pi^*g+h_{ab}\theta^a\otimes\theta^b.$$

The tensor γ has signature $(---+\underbrace{--\cdots-}_{n\ \text{times}})$. $h_{ab}=C^c_{ad}C^d_{cb}$ is a Killing tensor on G. This tensor has the form

$$\gamma_{AB}=\left(\begin{array}{c|c} g_{\alpha\beta} & 0 \\ \hline 0 & h_{ab}\end{array}\right). \tag{2.9}$$

It is clear that the frame θ^A is partially unholonomic, because

$$d\theta^a=\lambda(\Omega^a-\tfrac{1}{2}\lambda^{-2}C^a_{bc}\theta^b\wedge\theta^c)\neq 0. \tag{2.10}$$

We also introduce a dual frame

$$\gamma(\xi_A)=\gamma_{AB}\theta^B. \tag{2.11}$$

We have $\xi_A=(\xi_\alpha, \xi_a)$ and according to Trautman (1970)

$$\underset{\xi_a}{\mathscr{L}}\gamma=0. \tag{2.12}$$

Thus ξ_a are Killing vectors of metric γ.

Let us now define the Riemann connection ω_{AB} on P and covariant derivative D with respect to ω_{AB} (P is treated as a base of P'):

$$(D_\gamma)_{AB}=0 \qquad \text{and} \qquad D\theta^A=0. \tag{2.13}$$

The solution of (2.13) is

$$\begin{aligned}&\omega_{\alpha\beta}=\pi^*(\bar\omega_{\alpha\beta})-\tfrac{1}{2}\lambda H_{\alpha\beta a}\theta^a, \qquad \omega_{\alpha b}=-\omega_{b\alpha}=-\tfrac{1}{2}\lambda H_{\alpha\gamma b}\theta^\gamma,\\ &\omega_{ab}=-\omega_{ba}=-(2\lambda)^{-1}C_{abc}\theta^c.\end{aligned} \tag{2.14}$$

ω_{AB} is invariant with respect to the action of the group G (Trautman 1970). In the Klein–Kaluza theory $\lambda=2\varepsilon\sqrt{G}/c^2$, $\varepsilon^2=1$, where G is the gravitational constant and c is the velocity of light in a vacuum. This condition originated from consistency between equations in the Klein–Kaluza theory and Einstein's equation (Kaluza 1921, Lichnerowicz 1955a, Cho 1975). It is worth noting that this condition does not determine the sign of λ. It was unnoticed in Kalinowski (1981a, e).

Now we define the dual Cartan base on E. Let $\eta_{1234}=(-\det g)^{1/2}$ and $\eta_{\alpha\beta\gamma\delta}$ be a Levi–Civita symbol and

$$\eta_\alpha=\tfrac{1}{6}\bar\theta^\beta\wedge\bar\theta^\gamma\wedge\bar\theta^\delta\eta_{\alpha\beta\gamma\delta},\qquad \eta=\tfrac{1}{4}\bar\theta^\alpha\wedge\eta_\alpha. \tag{2.15}$$

Details concerning the elements of geometry mentioned here can be found in Trautman (1970, 1971, 1980), Kobayashi and Nomizu (1963) and Lichnerowicz (1955b).

3. $\frac{3}{2}$ spinors and dimensional reduction

Let us consider the group SO(1, $n+3$) and its fundamental (complex) representation of dimension $K=4\times 2^{[n/2]}$, where $[n/2]=l$ for $n=2l$ or $2l+1$

$$\begin{aligned}&U(g)\phi(X)=D^{\mathrm{F}}(g)\phi(g^{-1}X)\\ &X\in M^{(1,n+3)},\qquad g\in \mathrm{SO}(1,n+3).\end{aligned} \tag{3.1}$$

SO(1, $n+3$) acts linearly in $M^{(1,n+3)}$ (($n+4$)-dimensional Minkowski space).

Let Ψ be the one-form with values in spinor space of a fundamental representation of SO(1, n+3) (Spin(1, $n+3$)) defined on $M^{(1,3)}$ (Minkowski space). Thus we have

$$(\Psi,X)=\phi_X\in S^{\mathrm{F}}\qquad \text{(fundamental representation of Spin}(1,n+3)) \tag{3.2}$$

$X\in \mathrm{Tan}(M^{(1,3)})$. We call Ψ a one-spinor form.

After restriction of g to the subgroup SO(1, 3) we obtain a decomposition of D^{F} (Barut and Rączka 1977)

$$D^{\mathrm{F}}_{|\mathrm{SO}(1,3)}(\Lambda)=L(\Lambda)\underbrace{\oplus\cdots\oplus}_{2^{[n/2]}\text{ times}}L(\Lambda),\qquad \Lambda\in\mathrm{SO}(1,3), \tag{3.3}$$

where

$$L(\Lambda)=D^{(1/2,0)}(\Lambda)\oplus D^{(0,1/2)}(\Lambda).$$

is the Dirac representation of SO(1, 3).

The decomposition (3.2) of the one-spinor form Ψ has the form

$$\Psi_{|\mathrm{SO}(1,3)}=\begin{pmatrix}\psi_1\\ \psi\\ \vdots\\ \psi_{2^{[n/2]}}\end{pmatrix} \tag{3.4}$$

where ψ_i, $i=1,2,\ldots,2^{[n/2]}$ are one-spinor forms belonging to the Dirac representation ($L=D^{(1/2,0)}\oplus D^{(0,1/2)}$). Thus, due to the decomposition (3.3) we get a tower of one-spinor forms.

More precisely, we deal with representations of Spin(1, $n+3$) and Spin(1, 3) $\simeq$ SL(2, $\mathbb{C}$). Let us turn to a manifold P. It is a metric manifold (P, γ) with a metric tensor γ. At every point $p\in P$ a tangent space $T_p(P)\simeq M^{(1,n+3)}$. Let Ψ be a horizontal one-form spinor field defined on P. Then

$$(\Psi,X):P\to C^K,\qquad K=4\times 2^{[n/2]},\qquad (\Psi,\mathrm{ver}(X))=0,\qquad X\in\tan(P).$$

For the one-form spinor field Ψ we suppose the following action of the group G:

$$\Psi(pg_1)=\mathfrak{S}(g_1^{-1})\Psi(p) \tag{3.5}$$

where $p=(x,g)\in P$, $g, g_1\in \mathrm{G}$. $\mathfrak{S}$ is a representation of the group G in $4\times 2^{[n/2]}$-dimensional complex space, $x\in P$. If we take a section $e:E\to P$ we get a one-form spinor field $\Psi(e(x))$ on the manifold E (space–time). Thus at every point $x\in E$ we have after restriction to SO(1, 3) the one-spinor form $\Psi_{|\mathrm{SO}(1,3)}$, and for this the decomposition (3.4) is valid. Thus

$$(e^*\Psi)_{|\mathrm{SO}(1,3)}(x)=\begin{pmatrix}\psi_1(x)\\ \psi_2(x)\\ \vdots\\ \psi_{2^{[n/2]}}(x)\end{pmatrix}. \tag{3.6}$$

One-form spinor fields $\psi_i(x)$, $i=1,2,\ldots,2^{[n/2]}$ are one-form spinor fields at every point $x\in E$ belonging to the Dirac representation $L=D^{(0,1/2)}\oplus D^{(1/2,0)}$. We will call such a procedure the dimensional reduction for one-form spinor fields. In this way we have a tower of one-form spinor fields on E. The following graph symbolises this:

$$\Psi\xrightarrow[\text{section of } P]{e} e^*\Psi\xrightarrow[\substack{\text{from SO}(1,\,n+3)\\ \text{to SO}(1,\,3)}]{\text{restriction}}(e^*\Psi)_{|\mathrm{SO}(1,3)}\xrightarrow{\text{decomposition}}\begin{pmatrix}\psi_1\\ \psi_2\\ \vdots\\ \psi_{2^{[n/2]}}\end{pmatrix}.$$

In Kalinowski (1981a, 1982) we dealt in a similar context, with the five-dimensional (electromagnetic) case ($\mathrm{G}=\mathrm{U}(1)$, $n=1$). Thus we have the de Sitter group SO(1, 4), and we dealt with the one-form spinor Ψ belonging to the fundamental representation of the group $\mathrm{Spin}(1,4)\simeq\mathrm{Sp}(2,2)$. But for this case we have $\dim D^{\mathrm{F}}=\dim D^{\mathrm{F}}_{|\mathrm{SO}(1,3)}$ and after dimensional reduction we get only one one-form spinor field on E. The procedure (3.7) explains a construction given in Kalinowski (1981b, e). It is easy to see that

$$\Psi=\Psi_\mu\theta^\mu \qquad \text{and} \qquad \psi_i=\psi_{i\mu}\bar\theta^\mu, \qquad i=1,2,\ldots,w^{[n/2]}.$$

Thus the one-form ψ_i describes the $\frac{3}{2}$-spinor field on E (Deser and Zumino 1976), Ψ unifies a tower of one-forms ψ_i, i.e. a tower of $\frac{3}{2}$-spinor fields. (In Isenburg *et al* (1977) it was pointed out that $\frac{3}{2}$-spinor fields on E should be treated as one-form spinor fields.)

4. $\frac{3}{2}$-spinor field on P

Let Ψ be a one-form spinor field on P belonging to the fundamental representation D^{F} of SO(1, $n+3$) (Spin(1, $n+3$)), and let Γ^A, $A=1,2,\ldots,n+4$ be a representation of generators of the Clifford algebra for SO(1, $n+3$) acting in the space representation of D^{F} (i.e. $\Gamma^A\in \mathrm{C}(1,n+3)$)

$$\{\Gamma_A,\Gamma_B\}=2\bar g_{AB}, \qquad \Gamma^A\in\mathscr{L}(\mathbb{C}^k), \qquad K=4\times 2^{[n/2]}, \qquad [n/2]=l, \tag{4.1}$$

where $\bar g_{AB}=\mathrm{diag}(-1,-1,-1,1,\underbrace{-1\ldots-1}_{n\text{ times}})$. We introduce a one-form spinor field $\bar\Psi$:

$$\bar\Psi=\Psi^+\mathrm{B} \tag{4.2}$$

where + is Hermitian conjugation and

$$\Gamma^{\alpha+}=B\Gamma^\alpha B^{-1}. \tag{4.3}$$

It is easy to see that

$$\bar{\Psi}(pg_1) = \bar{\Psi}(p)\mathfrak{S}(g_1) \tag{4.4}$$

where $p \in (X, g) \in P$, $g, g_1 \in \mathrm{G}$, $\mathfrak{S}$ is a unitary representation of the group G acting in $4 \times 2^{[n/2]}$-dimensional complex space, $\mathfrak{S} \in \mathscr{L}(\mathbb{C}^k)$. The one-form fields Ψ and $\bar{\Psi}$ are defined on P and P is assumed to have an orthogonal coordinate system θ^A. This coordinate system is in general non-holonomic.

We perform an infinitesimal change of frame θ^A

$$\theta^{A'} = \theta^A + \delta\theta^A = \theta^A - \varepsilon^A_B\theta^B, \qquad \varepsilon_{AB} + \varepsilon_{BA} = 0. \tag{4.5}$$

Suppose that the field Ψ corresponds to θ^A and Ψ' to $\theta^{A'}$; then we obtain

$$\Psi' = \Psi + \delta\Psi = \Psi - \varepsilon^{AB}\hat{\mathfrak{S}}_{AB}\Psi, \qquad \bar{\Psi}' = \bar{\Psi} + \delta\bar{\Psi} = \bar{\Psi} + \bar{\Psi}\hat{\mathfrak{S}}_{AB}\varepsilon^{AB}, \tag{4.6}$$

where $\hat{\mathfrak{S}}_{AB} = \frac{1}{8}[\Gamma_A, \Gamma_B]$.

Now we consider covariant exterior derivatives of one form spinor fields Ψ and $\bar{\Psi}$ on P with respect to ω_{AB}. We obtain

$$\mathrm{D}\Psi = \mathrm{d}\Psi + \omega^{AB}\hat{\mathfrak{S}}_{AB} \wedge \Psi, \qquad \mathrm{D}\bar{\Psi} = \mathrm{d}\bar{\Psi} - \omega^{AB} \wedge \Psi\hat{\mathfrak{S}}_{AB}. \tag{4.7}$$

In Kalinowski (1981b, e) we introduced a new kind of 'gauge' derivative for the five-dimensional case. Now we generalise the approach to an arbitrary gauge group G:

$$\begin{aligned} \mathscr{D}\Psi &= \mathrm{hor}\, \mathrm{D}\Psi = \mathrm{d}_1\Psi + \mathrm{hor}(\omega^{AB})\hat{\mathfrak{S}}_{AB} \wedge \Psi, \\ \mathscr{D}\bar{\Psi} &= \mathrm{hor}\, \mathrm{D}\bar{\Psi} = \mathrm{d}_1\bar{\Psi} - \mathrm{hor}(\omega^{AB}) \wedge \bar{\Psi}\hat{\mathfrak{S}}_{AB}. \end{aligned} \tag{4.8}$$

Horizontality is understood in the sense of a connection ω on a bundle P.

Using (2.14) one obtains

$$\mathscr{D}\Psi = \bar{\mathscr{D}}\Psi - \tfrac{1}{8}\lambda H^{\alpha}{}_{\gamma}{}^{b}[\Gamma_\alpha, \Gamma_b]\theta^\gamma \wedge \Psi, \qquad \mathscr{D}\bar{\Psi} = \bar{\mathscr{D}}\bar{\Psi} + \tfrac{1}{8}\lambda H^{\alpha}{}_{\gamma}{}^{b}\theta^\gamma \wedge \bar{\Psi}[\Gamma_\alpha, \Gamma_b], \tag{4.9}$$

where

$$\bar{\mathscr{D}}\Psi = \mathrm{hor}\, \bar{\mathrm{D}}\Psi, \qquad \bar{\mathscr{D}}\bar{\Psi} = \mathrm{hor}\, \bar{\mathrm{D}}\bar{\Psi}. \tag{4.10}$$

$\bar{\mathrm{D}}$ is an exterior covariant derivative with respect to $\bar{\omega}_{\alpha\beta}$ (on E). $\bar{\mathscr{D}}$ is the normal gauge derivative and generally covariant derivative with respect to $\bar{\omega}_{\alpha\beta}$. It describes the well known minimal coupling scheme between the $\frac{3}{2}$-spinor field, gravitational field and Yang–Mills field.

It is easy to see that these new 'gauge' derivatives induce on P a new connection (P is understood as a base of P')

$$\hat{\omega}_{AB} = \mathrm{hor}(\omega_{AB}). \tag{4.11}$$

We work with $\hat{\omega}_{AB}$ rather than with ω_{AB}. In Kalinowski (1981b, e), due to these gauge derivatives one got the dipole electric moment of the fermion and avoided well known troubles (Thirring 1972). (Planck's mass term in Dirac's equation). The connection $\hat{\omega}_{AB}$ has many interesting features. In Kalinowski (1981d) it was proved that a scalar of curvature for $\hat{\omega}_{AB}$ is the sum of scalars of curvature for $\bar{\omega}_{\alpha\beta}$ (on E) and $-\frac{1}{4}\lambda^2 h_{ab}F^{a\mu\nu}F^b_{\mu\nu}$ (the Lagrangian of the Yang–Mills field for the gauge group G).

For ω_{AB} we get in addition an enormous cosmological term (Cho 1975). For the $\frac{3}{2}$-spinor field on E we have the Lagrangian four-form (Isenburg *et al* 1977, Kalinowski 1981b).

$$\mathscr{L}_{3/2}(\psi, \bar{\psi}, \mathrm{d}) = \tfrac{1}{2}\mathrm{i}\hbar c(\bar{\psi} \wedge \gamma_5\gamma \wedge \mathrm{d}\psi - \mathrm{d}\bar{\psi} \wedge \gamma_5\gamma \wedge \gamma \wedge) - \tfrac{1}{2}m\bar{\psi} \wedge \gamma_5\gamma \wedge \gamma \wedge \psi \tag{4.12}$$

where $\psi = \psi_\mu \bar\theta^\mu$, $\gamma_5\gamma = \gamma_5\gamma_\mu\bar\theta^\mu$, $\gamma = \gamma_\mu\bar\theta^\mu$, $\bar\psi = \bar\psi_\mu\bar\theta^\mu$. For ψ, $\bar\psi$ one supposes supplementary conditions:

$$l \wedge \psi = \bar\psi \wedge l = 0 \tag{4.12a}$$

where $l = \gamma^\mu \eta_\mu$.

Now we pass from ψ, $\bar\psi$ to Ψ, $\bar\Psi$ and from d to $\mathscr{D}$. In this way one generalises the minimal coupling scheme. (Classically we should pass from d to d_1.) Thus one easily writes

$$\mathscr{L}_{3/2}(\Psi, \bar\Psi, \mathscr{D}) = \tfrac{1}{2}\mathrm{i}\hbar c(\bar\Psi \wedge \Gamma_{2l+5}\Gamma \wedge \mathscr{D}\Psi - \mathscr{D}\bar\Psi \wedge \Gamma_{2l+5}\Gamma \wedge \Psi) - \tfrac{1}{2}m\bar\Psi \wedge \Gamma_{2l+5}\Gamma \wedge \Gamma \wedge \Psi \tag{4.13}$$

where $\Gamma = \Gamma_\mu\theta^\mu$, Γ_{2l+5}, $\Gamma = \Gamma_{2l+5}\Gamma_\mu\theta^\mu$, $\Psi = \Psi_\mu\theta^\mu$, $l = [n/2]$ (for details see appendix), and supplementary conditions

$$l \wedge \Psi = \bar\Psi \wedge l = 0 \tag{4.13a}$$

where $l = \Gamma_\mu \eta^\mu$.

Using (4.9) one easily gets

$$\mathscr{L}_{3/2}(\Psi, \bar\Psi, \mathscr{D}) = \mathscr{L}_{3/2}(\Psi, \bar\Psi, \bar{\mathscr{D}}) + \mathrm{i}(\sqrt{G}\hbar/4c)[H^{\alpha\nu b}\Psi_\lambda\Gamma_b[\Gamma_\alpha, \Gamma_\nu]\Psi^\lambda$$
$$+ 2H^{\alpha\rho b}(\bar\Psi_\lambda\Gamma_b\Gamma^\lambda\Gamma_\alpha\Psi_\rho + \bar\Psi_\rho\Gamma_b\Gamma^\lambda\Gamma_\alpha\Psi_\lambda - \bar\Psi_\lambda\Gamma^\lambda\Gamma_\rho\Gamma_b\Gamma^\nu\Gamma_\alpha\Psi_\nu)]\eta \tag{4.14}$$

and

$$2\sqrt{G\hbar}/c = 2 \times l_{\mathrm{pl}} q/\sqrt{\alpha} \simeq 0.95 \times 10^{-31}\ \mathrm{cm}(q) \tag{4.15}$$

where l_{pl} is Planck's length, α the fine structure constant, and q elementary charge.

If one performs the dimensional reduction (3.6) for $\mathscr{L}_{3/2}(\Psi, \bar\Psi, \bar{\mathscr{D}})$ one easily gets (see appendix)

$$\mathscr{L}_{3/2}(\Psi, \bar\Psi, \bar{\mathscr{D}}) \xrightarrow[\text{reduction}]{\text{dimensional}} \sum_{i=1}^{2^{[n/2]}} \mathscr{L}_{3/2}(\psi_i, \bar\psi_i, \bar{\mathscr{D}}). \tag{4.16}$$

Thus one obtains the interaction between the $\frac{3}{2}$-spinor fields ψ_i, $i = 1, 2, \ldots, 2^{[n/2]}$ and gravitation and Yang–Mills fields in a classical way already known. It is worth noticing that all $\frac{3}{2}$-fermions ψ_i have the same mass m.

Now we turn to the new term in (4.14). In Kalinowski (1981b) one deals with the five-dimensional (electromagnetic) case and interprets this term as the interaction of the electromagnetic field with a dipole electric moment for $\frac{3}{2}$-spinor fields of value $2\varepsilon l_{\mathrm{pl}} q/\sqrt{\alpha}$. Now we deal with Yang–Mills fields and should work with concrete useful representations of Γ^A. We will consider the cases $n = 2l$ and $n = 2l + 1$ separately.

If we suppose that the group G is a gauge group which unifies electromagnetic, weak and strong interactions, then G has a subgroup U(1) corresponding to electromagnetic interactions after breaking the symmetry. Let dim G $= 2l + 1$ and let a parameter of the electromagnetic subgroup U(1) correspond to $A = n + 4 = 2l + 5$. Then we turn to the additional term in the Lagrangian (4.14) and perform the dimensional reduction for $b = n + 4 = 2l + 5$.

One easily gets

$$\mathrm{i}(\sqrt{G}\hbar/4c)H^{\alpha\nu}{}_{2l+5}(\bar\Psi_\lambda\Gamma^{2l+5}[\Gamma_\alpha, \Gamma_\nu]\Psi^\lambda + 2\bar\Psi_\lambda\Gamma^{2l+5}\Gamma^\lambda\Gamma_\alpha\Psi_\nu + 2\bar\Psi_\lambda\Gamma^{2l+5}\Gamma^\lambda\Gamma_\alpha\Psi_\nu$$
$$+ 2\Psi_\nu\Gamma^{2l+5}\Gamma^\lambda\Gamma_\alpha\Psi_\lambda - \bar\Psi_\lambda\Gamma^\lambda\Gamma_\nu\Gamma^{2l+5}\Gamma^\rho\Gamma_\alpha\Psi_\rho)\eta$$

$$= 2\frac{l_{\rm pl}}{\sqrt{\alpha}}q\sum_{i=1}^{2^{[n/2]}} F^{\alpha\nu}{}_{2l+5}\bar{\psi}_{i\lambda}\gamma^5\mathfrak{S}_{\alpha\nu}\psi_i^\lambda$$

$$+\mathrm{i}(\sqrt{G}\hbar/2c)F^{\alpha\nu}{}_{2l+5}\sum_{i=1}^{2^{[n/2]}}(\bar{\psi}_{i\lambda}\gamma^5\gamma^\lambda\gamma_\alpha\psi_{i\nu}+\bar{\psi}_{i\lambda}\gamma^5\gamma^\lambda\gamma_\alpha\psi_{i\nu}$$

$$+\psi_{i\nu}\gamma^5\gamma^\lambda\gamma_\alpha\psi_{i\lambda}-\tfrac{1}{2}\bar{\psi}_{i\lambda}\gamma^\lambda\gamma_\nu\gamma^5\gamma^\rho\gamma_\alpha\psi_{i\rho})\eta \tag{4.17}$$

where $F^{\alpha\beta}{}_{2l+5}=F^{\alpha\beta}$ (electromagnetic field). Thus we get for all fermions a dipole electric moment of the order (4.15) as in Kalinowski 1981a, b, e). If $\dim \mathrm{G}=2l$ then this term is forbidden and we do not have the dipole electric moment of the fermion.

5. Rarita–Schwinger equation on manifold *P*

In this section we consider the generalised Rarita–Schwinger equation derived from (4.13). It is very well known that the minimal coupling scheme for $\frac{3}{2}$-spinor fields is inconsistent, for the Velo–Zwanziger (1969) paradox appears for the electromagnetic case. The Rarita–Schwinger equation in an external gravitational and electromagnetic field is relativistic covariant, but solutions are acausal. In this section we avoid these troubles due to generalised minimal coupling scheme described in § 4.

Now we derive the Euler–Lagrange equation starting from (4.13)

$$\mathrm{i}\hbar c\Gamma_{2l+5}\Gamma\wedge\mathscr{D}\Psi-\tfrac{1}{2}m\Gamma_{2l+5}\Gamma\wedge\Gamma\wedge\Psi=0 \tag{5.1}$$

and have the supplementary conditions (4.13*a*).

Using the definition of $\mathscr{D}$ (i.e. (4.9)) one easily gets

$$\mathrm{i}\Gamma_{2l+5}\Gamma\wedge\bar{\mathscr{D}}\Psi+\mathrm{i}\mathscr{H}[H^{\alpha\nu b}\Gamma_b[\Gamma_\alpha,\Gamma_\nu]\Psi^\lambda+2H^{\alpha\rho b}\Gamma_b\Gamma^\lambda\Gamma_\alpha\Psi_\rho+2H^{\alpha\lambda b}\Gamma_b\Gamma^\rho\Gamma_\alpha\Psi_\rho$$

$$-2H^{\alpha\rho b}\Gamma^\lambda\Gamma_\rho\Gamma^\nu\Gamma_\alpha\Gamma_b\Psi_\nu]\eta_\lambda-\frac{m}{2\hbar c}\Gamma_{2l+5}\Gamma\wedge\Gamma\wedge\Psi=0 \tag{5.2}$$

where

$$\mathscr{H}=\sqrt{G}/4c^2, \qquad l=[n/2]$$

and supplementary conditions (4.13*a*).

If $\mathscr{H}=0$ then (5.2) becomes the well known Rarita–Schwinger equation in an external gravitational and Yang–Mills field for (a tower) Ψ:

$$\mathrm{i}\Gamma_{2l+5}\Gamma\wedge\bar{\mathscr{D}}\Psi-(m/2\hbar c)\Gamma_{2l+5}\Gamma\wedge\Gamma\wedge\Psi=0. \tag{5.3}$$

Acting on both sides of (5.3) with $\bar{\mathscr{D}}$, one obtains

$$-\mathrm{i}\hbar c\Gamma_{2l+5}\Gamma\wedge(\Omega+\bar{\Omega}^{\alpha\beta}\mathfrak{S}_{\alpha\beta})\wedge\Psi+(m^2/4\hbar c)\Gamma_{2l+5}\Gamma\wedge\Gamma\wedge\Gamma\wedge\Psi=0,$$

where Ω is a curvature of a connection ω of the bundle P, $\bar{\Omega}^\alpha{}_\beta$ is a curvature of a connection $\bar{\omega}_{\alpha\beta}$ and $\bar{\Omega}^\alpha{}_\beta=\mathrm{d}\bar{\omega}^\alpha{}_\beta+\bar{\omega}^\alpha{}_\gamma\wedge\bar{\omega}^\gamma{}_\beta$, and supplementary conditions (4.13*a*).

Performing dimensional reduction, one obtains

$$-\mathrm{i}\hbar c\gamma_5\gamma\wedge(F+\bar{\Omega}^{\alpha\beta}\mathfrak{S}_{\alpha\beta})\psi_i+(m^2/4\hbar c)\gamma_5\gamma\wedge\gamma\wedge\gamma\wedge\psi_i=0,$$

$$i=1,2,\ldots,2^{[n/2]}, \tag{5.4}$$

where $F = e^*\Omega$ is a Yang–Mills field strength in a gauge e and supplementary conditions

$$l \wedge \psi_l = \bar{\psi}_i \wedge l = 0 \tag{5.5}$$

where $l = \gamma_\mu \eta^\mu$. Equations (5.3) and (5.4) are not differential equations. They are algebraic constraints for $\frac{3}{2}$-spinor fields. In this case, for every i, components of $\psi_{i\mu}$ are not independent. We must solve (5.2) modulo (5.3).

One expresses dependent components of Ψ_μ by independent ones and substitutes them into (5.2). Now the properties of (5.2) change drastically. The solutions become acausal (see Velo and Zwanziger 1969). This indicates that the classical minimal coupling schema is inconsistent for the Rarita–Schwinger equation.

Let us consider (5.1). Here we have the gauge derivative $\mathscr{D}$. Acting on both sides of (5.1) with $\mathscr{D}$ one gets, after some algebra,

$$\Gamma \wedge (\Omega + \bar{\Omega}^{\alpha\beta}\mathfrak{S}_{\alpha\beta}) \wedge \Psi + \tfrac{1}{2}\lambda \bar{\mathscr{D}} H^{\alpha b}\Gamma_b \wedge \Gamma\Gamma_\alpha \wedge \Psi + \tfrac{1}{2}\lambda\,\Omega^b \Gamma_b \wedge \bar{\mathscr{D}}\Psi$$
$$+ \tfrac{1}{16}\lambda^2 H^{\alpha b}\Gamma_b \wedge H^{\beta\alpha}\Gamma_a\Gamma_\alpha\Gamma\Gamma_\beta \wedge \Psi - (m^2/4\hbar^2c^2)\Gamma \wedge \Gamma \wedge \Gamma \wedge \Psi = 0 \tag{5.6}$$

where $H^{\alpha b} = H^{\alpha}{}_{\gamma}{}^{b}\theta^\gamma$.

Thus we get a differential equation for Ψ in place of the algebraic constraints (5.3) (iff $\Omega \neq 0$). If $\lambda \to 0$ one gets, from equation (5.6), equation (5.3). But this transformation is singular, for λ is a coefficient of the highest (first) derivative of Ψ in (4.6). In this way we avoid troubles with minimal coupling between gauge fields and the $\frac{3}{2}$-spinor field. Thus the Velo–Zwanziger paradox is absent. In the case of $m = 0$ we also avoid troubles with algebraic constraints obtained from the Rarita–Schwinger equation by differentiating.

6. Discrete transformations on *P*

Now let us consider operations of reflections defined on the manifold P. To perform these we choose a local coordinate system on P

$$X^A = (X^\alpha, X^a), \qquad X^\alpha = (\bar{X}, t).$$

Then

$$\Psi(p) = \Psi(X^A) = \Psi((\bar{X}, t), X^a). \tag{6.1}$$

We define transformations: space reflection Π, time reversal T, charge reflections C and combined transformations ΠC, $\theta = \Pi C T$ in the following way:

$$\Psi^C(X^\alpha, X^a) = \bar{C}\Psi^*(X^\alpha, -X^a) \tag{6.2}$$

where $\bar{C}^{-1}\Gamma_\mu \bar{C} = -\Gamma^*_\mu$. It is easy to see that

$$\bar{C} = \begin{pmatrix} 0 & & C \\ \vdots & \cdot^{\cdot^{\cdot}} & \vdots \\ C & & 0 \end{pmatrix} = C \otimes \prod_{i=1}^{[n/2]} \otimes \mathfrak{S}_1 \tag{6.3}$$

where $\mathfrak{S}_1$ is a Pauli matrix and C is an ordinary charge conjugation matrix on E (see appendix). Performing the dimensional reduction (3.6), one gets

$$\psi_i^C(X^\alpha) = C\psi_i^*(X^\alpha), \qquad i = 1, 2, \ldots, 2^{[n/2]} \tag{6.4}$$

or

$$\psi_{i\mu}(X^\alpha) = C\psi^*_{i\mu}(X^\alpha)$$

and all colour charges connected with the Yang–Mills field change sign.

In Kalinowski (1981a, e) and Thirring (1972) a similar problem was considered in the five-dimensional (electromagnetic) case. The reflection in the coordinate X^5 was interpreted as an electric charge conjugation (Rayski 1965). For the space coordinate reflection we have

$$\Psi^\Pi(X^\alpha, X^a) = \Gamma^4\Psi(-\bar{X}, t, X^a). \tag{6.5}$$

Performing the dimensional reduction (3.6), one gets (see appendix)

$$\psi^\Pi_i(\bar{X}, t) = \gamma^4\psi_i(-\bar{X}, t), \qquad \text{or} \qquad \psi^\pi_{i\mu}(\bar{X}, t) = \gamma^4\psi_{i\mu}(-\bar{X}, t),$$
$$i = 1, 2, \ldots, 2^{[n/2]}, \tag{6.6}$$

i.e. a normal parity operator on E. For the transformation of time reversal T we have

$$\Psi^T(\bar{X}, t, X^a) = \bar{C}^{-1}\Gamma^1\Gamma^2\Gamma^3\Psi^*(\bar{X} - t - X^a). \tag{6.7}$$

Performing the dimensional reduction (3.6), one gets (see appendix)

$$\psi^T_i(X, t) = C^{-1}\gamma^1\gamma^2\gamma^3\psi^*_i(\bar{X}, -t), \qquad i = 1, 2, \ldots, 2^{[n/2]},$$

or (6.8)

$$\psi^T_{i\mu}(\bar{X}, t) = C^{-1}\gamma^1\gamma^2\gamma^3\psi^*_{i\mu}(\bar{X}, -t)$$

and all colour charges connected with the Yang–Mills field of the gauge group G ($\dim G = n$) change sign, i.e. a normal time-reversal operator on space–time. For the transformation $\theta = \Pi CT$ we put

$$\Psi^\theta(\bar{X}, t, X^a) = -i\Gamma^{2l+5}\Psi(-\bar{X}, t, X^a) \tag{6.9}$$

where $l = [n/2]$ and $\Gamma^{2l+5} = \gamma^5 \otimes \prod_{i=1}^{[n/2]} \mathfrak{S}_1$ (see appendix). Performing the dimensional reduction (3.6), one gets

$$\psi^\theta_i(X, t) = -i\gamma^5\psi_i(-\bar{X}, t) \qquad \text{or} \qquad \psi^\theta_{i\mu}(X, t) = -i\gamma^5\psi_{i\mu}(-\bar{X}, t). \tag{6.10}$$

For the transformation ΠC one gets

$$\Psi^{\Pi C}(\bar{X}, t, X^a) = \Gamma^4\bar{C}\Psi^*(-\bar{X}, t, -X^a). \tag{6.11}$$

Performing the dimensional reduction, one gets

$$\psi^{\Pi C}_i(\bar{X}, t) = \gamma^4 C\psi^*_i(-\bar{X}, t) \qquad \text{or} \qquad \psi^{\Pi C}_{i\mu}(\bar{X}, t) = \gamma^4 C\psi^*_i(-\bar{X}, t),$$
$$i = 1, 2, \ldots, 2^{[n/2]}, \tag{6.12}$$

and all charges change sign.

It is clear that the transformations obtained here do not differ from those known from the literature. The additional term in the Lagrangian (4.4) (in the case $n = 2l + 1$) breaks the symmetry ΠC or T in an analogous way as in the five-dimensional case (Kalinowski 1981b) and for Dirac fields (Kalinowski 1981a, c, e). This can be easily seen by acting on the Lagrangian with the operator ΠC defined by (6.11).

Acknowledgment

I thank Professor A Trautman for his interest in the paper.

Appendix

In this appendix we deal with the Clifford algebra (Atiyah *et al* 1964, Cartan 1966), $C(1, n+3)$. Due to decomposition rules for $C(1, n+3)$ we write down a useful representation for Γ^A in terms of γ_μ.

It is well known that any Clifford algebra can be decomposed into a tensor product of the four elementary Clifford algebras (Atiyah *et al* 1964, Cartan 1966):

$$C(0,1) = \mathbb{C}, \text{ the complex numbers}, \qquad C(1,0) = R \oplus R,$$
$$C(0,2) = H = \text{quaternions}. \tag{A1}$$

We have

$$C(1, n+3) = C(0,2) \otimes C(1, n+1). \tag{A2}$$

Because we deal with dimensional reduction to the space–time E we define the Clifford algebra $C(1, 3)$ and we easily obtain

$$C(1, n+3) = \left(\prod_{i=1}^{[n/2]} \otimes C(0,2)\right) \otimes C(1,3) = \left(\prod_{i=1}^{[n/2]} \otimes H\right) \otimes C(1,3). \tag{A3}$$

It is also well known that either

$$C(1, n+3) = C(1, n+4) \qquad \text{iff } n+3 = 2l, \qquad l \in N_1^\infty$$

or

$$C(1, n+2) \simeq C(1, n+3) \qquad \text{iff } n+3 = 2l+1, \qquad l \in N_1^\infty. \tag{A4}$$

Let $\gamma_\mu \in \mathscr{L}(\mathbb{C}^4)$, $\mu = 1, 2, 3, 4$, be Dirac matrices obeying conventional relations

$$\{\gamma_\mu, \gamma_\nu\} = 2\eta_{\mu\nu}, \tag{A5}$$

$$\eta_{\mu\nu} = \operatorname{diag}(-1, -1, -1, +1), \qquad \gamma_5 = \gamma_1\gamma_2\gamma_3\gamma_4, \qquad \gamma_5^2 = -1, \tag{A6}$$

and let $\mathfrak{S}_i \in \mathscr{L}(\mathbb{C}^4)$, $i = 1, 2, 3$ be Pauli matrices obeying conventional relations as well:

$$\{\mathfrak{S}_i, \mathfrak{S}_j\} = 2\delta_{ij}, \tag{A7}$$

$$[\mathfrak{S}_i, \mathfrak{S}_j] = \varepsilon_{ijk}\mathfrak{S}_k. \tag{A8}$$

We also introduce the following notations: $\mathbb{1} \in \mathscr{L}(\mathbb{C}^2)$ is the 2×2 unit matrix and $\mathscr{I} \in \mathscr{L}(\mathbb{C}^4)$ is the 4×4 unit matrix. Thus one performs on the decomposition (A3) and easily gets

$$\Gamma^\mu = \gamma^\mu \otimes \left(\prod_{i=1}^{[n/2]} \otimes \mathfrak{S}_1\right) \tag{A9}$$

or

$$\Gamma^\mu = \begin{pmatrix} 0 & & \gamma^\mu \\ \vdots & \iddots & \vdots \\ \gamma^\mu & & 0 \end{pmatrix}. \tag{A10}$$

For $A \neq \mu$ one gets (in the case $n = 2l$),

$$\Gamma^{2p+1} = \mathrm{i}\mathscr{I} \otimes \left(\prod_{i=1}^{p-2} \otimes \mathbb{1} \right) \otimes \mathfrak{S}_3 \otimes \left(\prod_{i=1}^{l-p+1} \otimes \mathfrak{S}_1 \right)$$
$$\Gamma^{2p+2} = \mathrm{i}\mathscr{I} \otimes \left(\prod_{i=1}^{p-2} \otimes \mathbb{1} \right) \otimes \mathfrak{S}_2 \otimes \left(\prod_{i=1}^{l-p+1} \otimes \mathfrak{S}_1 \right), \tag{A11}$$

where $4 < 2p+1 < 2p+2 \leqslant n+4 = 2l+4$.

In the case $n = 2l$ we also define the matrix

$$\Gamma^{n+5} = ii^{3(l+1)} \prod_{A=1}^{n+4} \Gamma^A = (\gamma^5) \otimes \left(\prod_{i=1}^{l} \otimes \mathfrak{S}_1 \right) = \Gamma^{2l+5} \tag{A12}$$

or

$$\Gamma^{n+5} = \begin{pmatrix} 0 & \cdots & \gamma^5 \\ \vdots & \cdot^{\cdot^{\cdot}} & \vdots \\ \gamma^5 & & 0 \end{pmatrix} \tag{A13}$$

where $n = 2l$, $l \in N_1^{\infty}$.

If $n = 2l+1$ we have

$$\tilde{\Gamma}^A = \Gamma^A, \qquad A = 1, 2, \ldots, 2l+4$$
$$\tilde{\Gamma}^{n+4} = \Gamma^{2l+5} = \begin{pmatrix} 0 & \cdots & \gamma^5 \\ \vdots & \cdot^{\cdot^{\cdot}} & \vdots \\ \gamma^5 & & 0 \end{pmatrix}. \tag{A14}$$

It is easy to check that

$$(\Gamma^{2l+5})^2 = -1 \qquad \text{and} \qquad \{\tilde{\Gamma}^A, \Gamma^{2l+5}\} = 0 \qquad \text{for } A \neq 2l+5,, \tag{A15}$$

$$B = \bar{B} \otimes \left(\prod_{i=1}^{[n/2]} \otimes \mathfrak{S}_1 \right), \qquad \gamma^{\mu +} = \bar{B}\gamma^{\mu} B^{-1}. \tag{A16}$$

References

Atiyah M F, Bott R and Shapiro A 1964 *Clifford Modules, Topology Suppl. 1* 3

Barut O and Rączka R 1977 *Theory of group representations and applications* (Warsaw: PWN)

Cartan E 1966 *The theory of spinors* (Paris: Herman)

Cho Y 1975 *J. Math. Phys.* **16** 2029

Deser S and Zumino B 1976 *Phys. Lett.* **62B** 335

Isenburg I, Nester J M and Skinner R 1977 in *Proc. 8th Int. Conf. on General Relativity and Gravitation Waterloo, Canada* p 196

Kalinowski M W 1981a *Int. J. Theor. Phys.* **20** 563

—— 1981b $\frac{3}{2}$ *spinor field in the Klein–Kaluza theory, Warsaw University, preprint IFT/13/81*

—— 1981c *Spinor fields in nonabelian Klein–Kaluza theories, Warsaw University, preprint IFT/10/81*

—— 1981d *On Vanishing of the Cosmological Constant in Nonabelian Kaluza–Klein Theories Int. J. Theor. Phys.* in press

—— 1981e *Acta Phys. Austr.* **53** 229

Kaluza Th 1921, *Sitzungsber. Preuss. Akad. Wiss.* 966

Kobayashi S and Nomizu K 1963 *Foundations of differential geometry* vol I, II (New York: Interscience)

Lichnerowicz A 1955a *Théorie relativiste de la gravitation et de l'electromagnetisme* (Paris: Masson)

—— 1955b *Théorie global des connexions et de group d'holonomie* (Rome: Cremonese)

Rayski J 1965 *Acta Phys. Polon.* **18** 89
Thirring W 1972 *Acta Phys. Austr. Suppl.* 256
Trautman A 1970 *Rep. Math. Phys.* **1** 29
—— 1971 *Infinitesimal connections in physics, Lecture given at the Symp. New Mathematical Methods in Physics, Bonn, July 3, 1971*
—— 1980 *General Relativity and Gravitation* (*One hundred years after the Birth of Albert Einstein*) vol 1, ed A Held (New York: Plenum) p 287
Velo G and Zwanžiger D 1969 *Phys. Rev.* **186** 1337